Rational Order in Tone Scales and Cone Scales

The belief that nature must be considered as a standard from which art can derive its guidelines (*natura artis magistra*) was firmly established during many centuries. Not so firm were the reasons *why* art should apprentice itself to nature. The eighteenth century saw the transition from a neoclassical conception of nature as being regularly ordered, and therefore an example to mankind (as in Pope's *Nature methodized*) to the Romantic idea of man being overwhelmed by nature (following Burke's *delightful sublimity*). By showing two controversies in very different fields I intend to show how, in a more subtle way, also in other periods the idea of an *intrinsically rational order in nature* comes into conflict with a more practical, empirical attitude.

In his *Istituzioni Armoniche* of 1558, Italian musical theorist Giuseppe Zarlino proposed to consider not only octave, fifth and fourth, but also third and sixth as consonant intervals. Historically speaking, this correction on Pythagorean thinking was long overdue. Thirds and sixths had gradually come to be accepted as harmonic shelters since the earliest forms of polyphony came into existence. But not before Zarlino did the major third acquire the prestigious position of being one of the cornerstones of the harmonic framework.

Zarlino's correction marks the end of the predominance of the Pythagorean *tetraktys* as a theoretical basis for harmony: the *tetraktys* allows only those intervals as consonant whose ratios can be expressed by the first four numbers.[1] Zarlino introduces a new concept in music theory: the *senario*, implying that *six* rather than *four* is the limit for the ratios that build up consonant intervals. Enter the major third (5 : 4), the minor third (6 : 5) and their counterparts, the minor sixth (8 : 5, where 8 is considered the twofold of 4) and the major sixth (5 : 3). The Venetian maestro believed that *just intonation* could be achieved by basing all intervals in a tone scale on the fifth and the major third. That leads to the only type of intonation which Zarlino is willing to consider as *natural*.[2] In other words: Zarlino did not so

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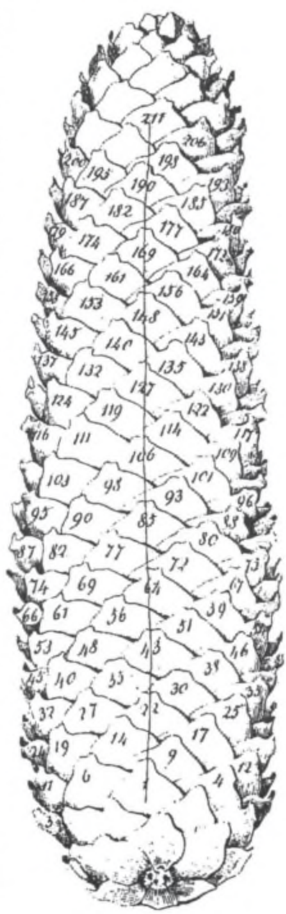
[1] That is: the octave (2 : 1), the fifth (3 : 2) and the fourth (4 : 3) – and, trivially, the prime (1 : 1).

[2] *Just* and *natural* are still in use as synonyms for this particular intonation (in German: *reine* or *natürliche Stimmung*).

much *overthrow* the Pythagorean way of thinking in terms of rational order based on numerical ontology, but rather *saved* it by extending the range of fundamental numbers to six.

The attack on the ontological basis of this type of thinking was left to Vincenzo Galilei, father of the famous astronomer but also a pupil of Zarlino's. Galilei does not accept his master's guideline of the *senario.* In particular, he attacks the status of fifth and third as »natural« intervals. *No such thing* – says Galilei: all intervals, all tone scales have come to be established by human convention. Exact rational proportions (in the mathematical sense of being expressible as a ratio of integers) have no special meaning here. There is no principal difference, in this respect, between the intervals of music and the words of a »natural« language.

Galilei's critical attitude towards his master's authority is fundamental. The idea that a consonant interval should be anything else but a rational number would have been considered absurd during the major part of European history. The foundation of that thought goes back at least as far as Plato's *Timaeus,* where the very ratios of the *tetraktys* are constitutive for the created order of the cosmos. Galilei's criticism clearly reflects more than just a musicological comment; it heralds the paradigm shift with which the name of his son will forever be linked. But before going deeper into the heated debate between master and pupil, we shall first take a look at a conflict in a completely different setting and time – not about a human product, but concerning the production of nature herself.



Towards the middle of the nineteenth century, a botanical debate flared up about the way in which nature accomodates certain primordia around a centre – like leaves around a stem, scales on a pine cone, sunflower seeds on a flower head, etc. Though we use to wonder about the amazing spiral structures which these plants show, we often do not realize that these spirals were not there in the first place. They come into existence step by step; in fact, the birth certificates of all the sunflower seeds are issued one by one, in a strict order that can even be traced subsequently. The spirals we see are no more than an epiphenomenon of a spiral we don«t see, but which we can obtain by connecting the scales in the order in which they popped up. We shall call this the *fundamental spiral* (in the picture: 1-2-3-4 etc.), whereas the contiguous parallels as they become visible are called *parastichies* (in the picture: 6-14-22-30, or 19-27-35-43 etc.).

By 1830, German botanist Alexander Braun had the brilliant idea to use the precise order of these scales for the classification of coniferous plants.[3] Classification being a favourite pastime for botanists, the subtle differences between the implantation of the scales in the different species of coniferous plants seemed to offer an ideal handle to come to grips with the differences between them, and to label these differences. In order to work out these labels, Braun introduced the notion of *divergence* in botanical parlance. By notating such a divergence as, say, 8∫21 (as in the case of the pine cone on the picture), Braun meant that 21 scales were found when the fundamental spiral had rounded the cone exactly 8 times.[4]

The presupposition of this project is that the position of (in this case) the 22nd scale is exactly above the first. Braun's conception implies that, apart from the parastichies, each cone also shows parallel *orthostichies* (in the picture: 1-22-43-64, or 9-30-51-72 etc.). Braun does indeed believe that after a natural number of scales the fundamental spiral has come full circle, so that the ratio of the number of scales and the number of rotations can be expressed as an exact rational number.

No such thing – say two French scientists who started investigating coniferous plants around the same time as Braun did. Auguste and Louis Bravais observe the same cones as Braun, but see something entirely different. In particular, they do not see a series of distinctly different ratios in the divergences of the plants. Braun's differentiation is but an illusion, or so they claim. Nature has found the optimum angle for the implantation of every next seed or scale; that angle ensures that all the primordia have an optimum space to grow, and it remains the same at every turn: 137° 30' 28".[5] That amounts to a repeated division of the circle according to the golden section, which is an irrational measure and can, for that reason, never lead to the rational classification that Braun pursued. It is, however, a *constant* measure – the only one that grants equal rights to all primordia. The whole organ-

[3] A. Braun, »Vergleichende Untersuchung über die Ordnung der Schuppen an den Tannenzapfen als Einleitung zur Untersuchung der Blattstellung überhaupt«, in *Nova Acta Academicae Caesareae Germanicae Leopoldinae,* Nr. 15, 1830, pp. 199-401; reprinted in book form in Bonn, 1831. Page numbers in this article refer to the book edition.

[4] Numerator and denominator of the divergence will generally relate as the numbers $(n-1):(n+1)$ from the Fibonacci series 1, 1, 2, 3, 5, 8, 13, 21, 34 ....

[5] L. & A. Bravais, »Essai sur la disposition des feuilles curvisériés«, in *Annales des Sciences Naturelles,* Seconde Série, t. 8ème, 1837, pp. 70/1.

ism benefits from this equal division. Recent research[6] has shown that this is, in fact, the way nature behaves; one does not need to involve genetical or teleological principles to find that the flower head of a sunflower is divided again and again, by each new primordium, according to the golden section.

Both controversies, the one in the Renaissance about the alleged rationality of tone scales and the one in the nineteenth century about the alleged rationality of cone scales, find their origin in opposing conceptions of the value of rational order in nature. Of course, both pairs of opponents have a lot in common, due to the preconceptions that even opponents would share in a certain age. Both Zarlino and Galilei frequently call on »the ancients« to substantiate their own point of view; both believe that the ancients had set an example, not so much by their high standard of cultural development, but by their being closer to *nature*, that is, by their better understanding of *natural order*.

Zarlino believes that Mother Nature restricts herself to a well-considered dose of perfection by differentiating between the individuals that belong to the same species rather than just cloning the ideal archetype again and again. He praises the ancients for transposing that principle to music, where repetition of identical consonant intervals is to be avoided:

»Thus they held it as true that whenever one had arrived at perfect consonance one had attained the end and the perfection toward which music tends, and in order not to give the ear too much of this perfection they did not wish it repeated over and over again.

The truth and excellence of this admirable and useful admonition are confirmed by the operations of Nature, for in bringing into being the individuals of each species she makes them similar to one another in general, yet different in some particular, a difference or variety affording much pleasure to our senses. This admirable order the composer ought to imitate, for the more his operations resemble those of our great mother, the more he will be esteemed. And to this course the numbers and proportions invite him, for in their natural order one will not find two similar proportions following one another immediately ....«[7]

Vincenzo Galilei is involved in a different battle. He is a member of the Florentine *Camerata*, the think-tank of humanist scholars and noblemen who paved the way for an entirely new form of art, a spectacle that would con-

[6] S. Douady & Y. Couder, »Phyllotaxis as a Physical Self-Organized Growth Pattern«, in *Physical Review Letters*, Vol. 68, Nr. 13, 1992, pp. 2098-2101.

[7] G. Zarlino, *Istituzioni Armoniche*, in O. Strunk (ed.), *Source readings in Music History, Vol. II – The Renaissance*, New York/London 1965, pp. 44/5.

quer European stages in the seventeenth century: opera. Opera is typically an art form that did not result directly from any development in musical practice, but was prepared on the drawing board. The main impulse came from the Florentine resistance against contemporary (»modern«) polyphony. Galilei's *Dialogo della musica antica e della moderna*[8] is an ardent plea for a new type of music (»postmodern«, so to speak), that would do justice to the natural expression of human affections – a task which polyphonic music, with its intricate structure of simultaneous melodies giving voice to several texts at the same moment, could not possibly fulfil. The polyphonic music of Galilei's contemporaries is an insult to human nature (so he believes), and the music of antiquity is put forward in his writings as an inspiring guideline.

Intrinsically, differences of opinion between Zarlino and Galilei are not as great as their personal feud might suggest. Galilei would have no trouble with the quotation given above, regarding the desired variety in intervals, and Zarlino would wholeheartedly agree with the Camerata's preference for words above melody when putting text to music. Those were in fact the central issues of the time, and both authors were well aware of them. But unfortunately, both men were driven by ».... the desperate wish to contradict each other«.[9] The advantage of this for later scholars is that their different attitudes towards the importance of rational order received much emphasis, and thus clearly expose the difference between Zarlino's neoplatonism and Galilei's more empirical approach.

Empirical research, as it became to be practised by the investigative Renaissance minds, did not automatically imply a repudiation of rational proportion. Galilei made a name for himself in the history of music theory by correcting what the Middle Ages had believed was an observation by Pythagoras himself: the discovery of the proportional relationships between the weight of the hammers used by the blacksmith, and the pitches of the sounds they produced. Every medieval music theorist knew that if a certain pitch was produced by tying a weight to a string, the octave of that pitch would be produced by tying the double weight to the same string, and a fifth with the help of a weight one and a half times the original, etc. In other words: these ratios were supposed to be the simple inversion of the (more easily measurable) ratios for string lengths producing the same intervals. Not so, says Galilei: to produce those intervals by tension, the weights would have to be in *squared* inverse proportion to the lengths of the strings. Their relationships to the perfect consonant intervals are still perfectly expressible as

8 Florence, 1581.

9 D.P. Walker, *Studies in Musical Science in the Late Renaissance*, Leiden 1978, p. 16.

ratios of whole numbers, but not anymore in the traditionally constitutive numbers of the Pythagorean *tetraktys*.

How did Galilei find this out? Going by his repeated reference to experimental method (*con il mezzo dell«esperienza*), we may safely assume: by trying out.

Zarlino, as we saw, did not stick either to the *tetraktys* to express the ratios of the imperfect consonances, but his argumentational back-up is of a totally different order. Why should the *senario* rather than the *tetraktys* be considered as the basis for our harmonic understanding? As if we could not have guessed:

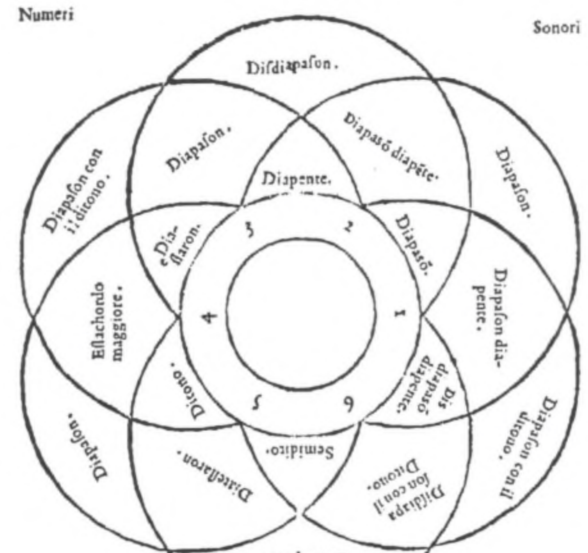
– God created the world in six days

– six signs of the zodiac are always above the earth, the other six are invisible

– there are six »planets« (to Zarlino's knowledge: Saturn, Jupiter, Mars, Venus, Mercury, and the moon)

– there are six directions (up, down, ahead, behind, left, and right; Zarlino calls on Plato to testify to this spatial insight)

– the number 6 is traditionally hailed as the first »perfect number«; that is, it equals the sum of its dividends 1, 2 and 3; moreover, it is their product



– in music, there are six »authentic« and six »plagal« modes.

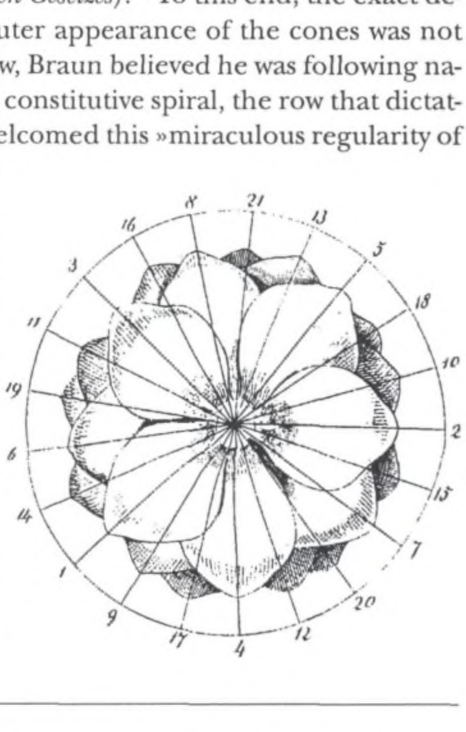
Zarlino gives quite a few more reasons,[10] but these six will suffice to show the gap that extends between the mental world of Zarlino and that of his pupil. Galilei, who was an early pioneer of *equal temperament*, did not feel anything was lost by giving up the perfectly rationally ordered intervals. Zarlino, on the other hand, could not imagine just intonation

[10] See C.V. Palisca, *Humanism in Italian Renaissance Musical Thought*, New Haven/London 1985, p. 248.

in any other way than by the *numeri sonori* of the *senario*, as this illustration from his book shows: a well-ordered world of musical intervals, with the *senario* in the centre.

The controversy between Alexander Braun and the Bravais brothers is situated in a different age, against the background of different scientific strategies. Experimental verification had become part and parcel of regular scientific behaviour by the time Braun developed his theory, and he himself was no exception: thousands of pine cones were collected by him and his colleague, Carl Schimper, and meticulously sorted out and classified. And yet, Braun is steered by another drive than collector's mania or labelling neurosis: he wants to unravel the hidden principle behind natural order as this becomes visible in the arrangement of leaves, seeds, petals and scales along a stem.

What Braun finds is fascinating, but much more fascinating is to know what he is looking for. Braun was, in his own words, chasing the »joyful presumption of a law founded deeply in the life of the plants« (*freudige Ahnung eines tief im Leben der Pflanze gegründeten Gesetzes*).[11] To this end, the exact description and classification of the outer appearance of the cones was not enough. In looking for his hidden law, Braun believed he was following nature herself. And when he found the constitutive spiral, the row that dictated the position of all the scales, he welcomed this »miraculous regularity of order« (*wunderbare Gesetzmässigkeit der Anordnung*) with an almost religious respect: »In this last, One Row, dawning upon our expectation, we behold the true goal of our hope, the One Ground of phyllotaxis, on which all multitude and variety of rows must rest.«[12]



Braun's drawing, within a circle, of a bottom view of the pine cone shows one layer of this rational order. It is almost reminiscent of the picture in Zarlino's book: a rounded way of thinking that always comes back to its point of departure.

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[11] *Vergleichende Untersuchung*, p. 3.

[12] »In dieser uns in der Erwartung vorschwebenden letzten, Einen Reihe erblicken wir das wahre Ziel unserer Hoffnung, den Einen Grund der Blattstellung, auf dem alle Vielheit und Vielartigkeit der Reihen beruhen muss.« *Vergleichende Untersuchung*, p. 22.

There is an intriguing tension between unity and variety in Braun's conception of natural order, comparable to the way Zarlino deals with the perfection of consonants and their necessary differentiation in musical composition. The unity that is firmly established in the overall ruling of the fundamental spiral serves as a condition to bring out a multitude of differences – differences by which the several species of cones can be distinguished and labelled. Braun's aim is a classification in the line of Linnaeus, arrived at by means of empirical observation, but his regulative conception is that of an overall rational order. In other words: Braun treats divergences as if they were musical intervals according to a traditional system of temperament, and he does so on the basis of a deeply rooted inner conviction that this is how nature behaves. Braun's phyllotaxis reflects an order of *just intonation.*

It is to this preconception that the Bravais brothers oppose. There *is* no discrete classification of different divergences; when trying to attribute one of Braun's rational labels to a specific plant, the choice between, say, 8∫21 or 13∫34 often seems quite arbitrary. None of Braun's alleged observations is as precise as the exactitude of the rational measure suggests. The brothers carefully justify this statement with a number of illustrations. What they object to is in fact not so much the validity of Braun's equally careful observations, but the very status of the starting point which led these observations to result in the conclusions that Braun presented. That starting point is the concept of *orthostichy*, which, to continue the metaphor I have just introduced, in Braun's system of just botanical intonation fulfils the role of the *octave,* the point of reference for all the other intervals. The strong impact of Braun's conception becomes clear when we read that Carl Friedrich Naumann considered the orthostichy as »the real essence« (*das eigentliche Wesen*) and parastichies as »a mere phenomenon of phyllotaxis« (*ein blosses Phänomen der Blattstellung*).[13] The alternative which the Bravais brothers present comes down to granting identical rights to primordia in the same way tones have identical rights in equal temperature – with the proviso that in the case of the plants, this equality is granted by nature.

Apart from carefully explaining their own theory, the Bravais brothers make a stand against Braun's position in a separate article.[14] The tone of this article is (as opposed to Galilei's tone towards Zarlino) mild and respectful; Braun and Schimper are given ample credit for their research, and the opposition against the notion of *orthostichy* is very carefully presented. Braun

[13] C.F. Naumann, *Über den Quincunx als Grundgesetz der Blattstellung vieler Pflanzen*, Dresden/Leipzig 1845, Vorwort.

[14] Attached to the German translation of their work: L. & A. Bravais, *Über die geometrische Anordnung der Blätter und der Blüthenstände*, Breslau 1839.

is less attentive in his reply to the brothers in a later book.[15] In order to counterdict the French criticism, Braun tries to find a theoretical peg from an area where rational order had come to be understood and generally accepted: crystallography. As this happened to be Auguste Bravais's field of expertise, and as he had even been one of the pioneers in establishing which classes of crystals were morphologically possible, Braun seems to beat his opponent at his own game when he claims that wiping out the differences between the several rational divergences would amount to saying that all crystal forms are not really different because they all have the sphere as their limit.[16]

This argument sounds stronger than it is. Crystal forms are different for constructive reasons; as opposed to phyllotaxis, each specific form is the result of a different chemical build-up that is discretely established from the beginning. Whatever possibilities there are, the sphere is not among them. But it is an excellent illustration of Braun's way of thinking. He wants to see his covering law as a regulative principle, not as a generalisation of empirical data. Transcendent unity must appear to the senses as phenomenal variety. The realm of truth is not to be found in experience, but in the mind: »All truth is mental«, says Braun; »all facts become recognized truths only when we can mentally construct them«.[17]

It is almost touching to read how Nees von Esenbeck, the author of the introduction to the German translation of the Bravais writings, tries to unite the contribution of both parties in one encompassing reconciliation: having made clear that it was his compatriots Braun and Schimper who led the way and who took care of the essential discoveries, he compliments the Bravais brothers for their mathematical fine-tuning of the issue. The discovery of the »essentially irrational proportion« (*das wesentlich irrationale Verhältniss*) involved in the divergences, leads in his eye to the »ideal infinity of the fundamental spiral« (*die ideale Unendlichkeit der Grundwendel*). And he continues: »both these significant results are redeeming features not only for the metamorphosis of plants, but indeed for the philosophical contemplation of the organized world. It confirms the conviction that even the originally rational arrangements of leaves are subjected to the fundamental law of irrational [phyllotaxis], and are recognized as mere multiples of them«.[18]

[15] A. Braun, *Betrachtungen über die Erscheinung der Verjüngung in der Natur*, Leipzig 1851.

[16] *Betrachtungen*, p. 126.

[17] A. Braun, »Dr. Carl Schimper's »Vorträge über die Möglichkeit eines wissenschaftlichen Verständnisses der Blattstellung«, in *Flora*, Jg. 18, I. Band, 1835, p. 146.

[18] There seems to be a word lacking in the German text; maybe the dash after *irrationalen* in the manuscript was meant to repeat *Blattstellungen*: »diese beiden bedeutsamen Resultate sind Lichtpuncte nicht allein für die Pflanzenmetamorphose, sondern für die philosophische Betrachtung der organisirten Welt überhaupt. Man sieht mit

This is a surprising point of view. It combines the mathematical conclusion of the Bravais brothers concerning phyllotaxis with Braun's philosophical idealism concerning the fundamental order that prevails in nature – and yet manages to squeeze in the idea that these arrangements are »originally rational«.

It is not very difficult to round off empirical data concerning musical intervals or botanical primordia in such a way that the rationality hypothesis is confirmed. Both tone scales and cone scales come very close indeed. But this rationality comes about as a result of human evaluation. Whether, in the end, nature does or doesn«t show rational order, depends – not on the nature of nature, but on the nature of our conception of natural order.

verstärkter Ueberzeugung, wie selbst die ursprünglich rationalen Blattstellungen der Pflanzen sich dem Grundgesetze der irrationalen – unterordnen, und als blosse Vielfache derselben erkannt werden (....).« L. & A. Bravais, *Über die geometrische Anordnung der Blätter und der Blüthenstände*, Breslau 1839, pp. V/VI.